Math 4200
Monday October 26
3.2 Power series and Taylor series for analytic functions. We begin with the last example from Friday, which we did not get to.

Announcements:

Warm-up exercise : We showed on Friday that

$$* \quad \frac{1}{1-z} = 1 + z + z^2 + \dots = \sum_{n=0}^{\infty} z^n \qquad \text{for } |z|<1$$

• (and series diverges for $|z| \geq 1$)

Can you use substitution to find a series for

$$** \quad \frac{1}{z-3} = \frac{1}{-3(1-z/3)} = -\frac{1}{3}\underbrace{\frac{1}{1-z/3}}$$

that converges for $|z|<3$?

subst $z/3$ into 1st series

$$= -\frac{1}{3}\sum_{n=0}^{\infty} \frac{z^n}{3^n}$$

converge for $|z/3|<1$
$(|z|<3)$
diverge for $|z| \geq 3$.

$z \quad + \quad \frac{z^2}{2!}$

$\underbrace{f'(z) = 0 + 1 + \frac{2z}{2!}} + \frac{3z^2}{3!} + \cdots$

Warm-up and summary/intro problem:

$$e^z \overset{?}{=} f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \ldots.$$

$f(0) = 1$

(1) converges uniformly absolutely for $|z| \leq R$, so converges to an analytic function $\forall\ z$. Then use the term by term differentiation theorem to show that $f'(z) = f(z)$ (2) and use this and $f(0) = 1$ to identify $f(z)$. (3)

(1) Weierstrasse M test.

$|z| \leq R$, then $\sum_{n=0}^{\infty} \frac{|z|^n}{n!} \leq \sum_{n=0}^{\infty} \frac{R^n}{n!} \xrightarrow{\text{recall}} = e^R$

$M_n = \frac{R^n}{n!}$ on $\overline{D(0;R)}$

OR

use ratio test.

$$\frac{\frac{R^{n+1}}{(n+1)!}}{\frac{R^n}{n!}} = R\frac{n!}{(n+1)!} = \frac{R}{n+1}$$

$$\lim_{n\to\infty} \frac{R}{n+1} = 0 < 1$$

so series conv. by ratio test.

$$f'(z) = \sum_{n=1}^{\infty} \frac{d}{dz} \frac{z^n}{n!}$$

(2) $$f'(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = f(z)$$

$n-1=k$

(3) $$f'(z) - f(z) = 0$$

$$e^{-z}\left(f'(z) - f(z)\right) = 0$$

$$\frac{d}{dz}\left(e^{-z} f(z)\right) \equiv 0$$

$$e^{-z} f(z) = C$$

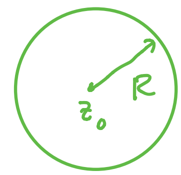
$$f(z) = Ce^z$$

$$f(0) = 1 = Ce^0 \Rightarrow C = 1$$

$$\boxed{f(z) = e^z}$$

Power series

Consider the *power series* centered at z_0



$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad a_n, z_0 \in \mathbb{C}.$$

Theorem 1:

(i) There exists unique $R \in [0, \infty]$ such that the power series above converges unif absolutely $\forall\ z$ with $|z - z_0| < R$ and diverges for all z with $|z - z_0| > R$. This value of R is called the *radius of convergence* of the power series.

(ii) For $r < R$, the convergence of the power series is uniform absolute convergence $\forall\ z \in \mathrm{D}(z_0, r)$. Thus f is analytic in $\mathrm{D}(z_0\,; R)$.

proof: Notice that the radius of convergence (if it exists) is uniquely determined by the two conditions it must satisfy. Define $R := sup\{r \geq 0 \mid \sum_{n=0}^{\infty} |a_n|\, r^n < \infty\}$. We'll show that this number R satisfies the two required contitions so it will be the radius of convergence. It is either a non-negative real number or $+\infty$:

Let $r < R$ and apply the Weierestrass (1) M test in $\overline{\mathrm{D}}(z_0, r)$ to deduce uniform absolute convergence of the power series for $f(z)$ in $\mathrm{D}(z_0, r)$. Thus the power series converges in $\mathrm{D}(z_0, R)$ to an analytic function and we have shown (ii), and half of (i).

(1) $\left\{ r \;\middle|\; \sum_{n=0}^{\infty} |a_n| r^n < \infty \right\}$ is either $\{0\}$
$r \geq 0$
or it's an interval where left endpt is zero

$r_1 < r_2$

$\sum |a_n| r_1^n \leq \sum |a_n| r_2^n$

On $\overline{D(z_0, r)}$ use W.M. test

$|z - z_0| \leq r$. $\quad \sum_{n=0}^{\infty} |a_n| |z - z_0|^n \leq \sum_{n=0}^{\infty} \underbrace{|a_n| r^n}_{M_n.} < \infty$ by def of R and (1).

Then show the second part of (ii), i.e. divergence when $|z - z_0| > R$, by proving and using

Abel's Lemma: If $sup_{n \in \mathbb{N}} \{|a_n| R_1^n\} = M < \infty$ then $\sum_{n=0}^{\infty} |a_n| r^n < \infty$ * $\forall\ 0 < r < R_1$

Suppose $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges.

$\Rightarrow |a_n (z-z_0)^n| \to 0$ as $n \to \infty$ *

$\Rightarrow \exists M$ s.t. terms unif bd.

By Abels lemma.

$\sum |a_n| r^n < \infty$

$\forall\ 0 < r < |z - z_0|$

recall $R = \sup\{r \text{ s.t. } \sum |a_n| r^n < \infty\}$.

$\Rightarrow |z - z_0| \leq R$.

i.e. $|z - z_0| > R$ series diverges.

$\blacksquare$

prove Abel!

Let $0 < r < R_1$

$$\sum_{n=0}^{\infty} |a_n| r^n = \sum_{n=0}^{\infty} \underbrace{|a_n| R_1^n}_{\leq M} \left(\frac{r}{R_1}\right)^n \qquad \frac{r}{R_1} < 1$$

$$= M \sum_{n=0}^{\infty} \left(\frac{r}{R_1}\right)^n$$

$$= M \frac{1}{1 - \frac{r}{R_1}} < \infty.$$

$\blacksquare$

Theorem 2 (differentiation and integration of power series) Consider

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad a_n, z_0 \in \mathbb{C}$$

$|z-z_0|<R$

with radius of convergence $R > 0$. Then $f'(z)$ can be computed via term by term differentiation; and the antiderivatives $F(z)$ of $f(z)$ can be computed by term by term antidifferentiation (plus an additive constant). The power series for f' and F have the same radius of convergence R as does the power series f.

- $$f'(z) = \sum_{n=1}^{\infty} n\, a_n (z - z_0)^{n-1} \qquad \forall\, z \in D(z_0, R)$$
- $$F(z) = F(z_0) + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1} \qquad \forall\, z \in D(z_0, R)$$

proof: Since the power series for $f(z)$ converges in $D(z_0, R)$, and uniformly absolutely for any closed subdisk (concentric or not, since each closed subdisk is contained in a closed concentric sub-disk), we deduce from Theorem B′ Friday that the term-by-term differentiated power series for $f'(z)$ also converges in $D(z_0, R)$, and uniformly for any closed subdisk. Thus the radius of convergence for the $f'(z)$ series is at least the radius of convergence for the f series. But using the characterization of radius of convergence from Theorem 1, the radius convergence for the series for $f'(z)$ is at most R, since the moduli of the terms in the f' series are larger than in the f series:

rad of conv for f' rad of conv for f

$$\sup\{r \geq 0 \mid \sum_{n=1}^{\infty} n|a_n|\, r^n < \infty\} \leq \sup\{r \geq 0 \mid \sum_{n=0}^{\infty} |a_n|\, r^n < \infty\} = R.$$

if this converges then this converges

Thus the radii of convergence for f, f' must be the same. Thus also the radius of convergence for $F, F' = f$ must be equal.

QED.

Theorem 3. (Uniqueness of power series representations) If f is given by a power series

$$\boxed{f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n} \quad \Rightarrow f(z_0)=a_0 \qquad a_n,\, z_0 \in \mathbb{C} \qquad a_n,\, z_0 \in \mathbb{C}$$

with positive radius of convergence R then the power series *is* the Taylor series with

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad n = 0, 1, 2, \ldots$$

In particular, a given analytic function whose domain of analyticity includes z_0 can have only one power series representation centered at z_0.

proof: We know from the previous Theorem that we have

$$\boxed{f'(z)=\sum_{n=1}^{\infty} n\, a_n (z-z_0)^{n-1}} \qquad \forall\, z \in \mathrm{D}(z_0, R)$$

$f'(z_0) = 1\cdot a_1 + 0 + 0 \qquad a_1 = f'(z_0)$

and inductively, for $k \in \mathbb{N}$,

$$f^{(k)}(z)=\sum_{n=k}^{\infty} \frac{n!}{\underbrace{(n-k)!}_{n(n-1)(n-2)\cdots(n-(k-1))}}\, a_n (z-z_0)^{n-k} \qquad \forall\, z \in \mathrm{D}(z_0, R),\ \forall\, k \in \mathbb{N}.$$

$f^{(k)}(z_0) = k(k-1)\cdots 2\cdot 1\ a_k$

evaluating at z_0 only the first term in the series is nonzero, so

$$f^{(k)}(z_0) = k!\, a_k \quad \Rightarrow a_k = \frac{f^{(k)}(z_0)}{k!}.$$

Theorem 4 If f is analytic in $\mathrm{D}(z_0; R_1)$ then the Taylor series for f at z_0,

$$f(z)=\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$

converges to f in $\mathrm{D}(z_0; R_1)$. Thus the radius of convergence of the Taylor series is at least R_1. And, one can use this to get an upper bound on the radius of convergence: if $\exists\, z_1$ such that f cannot be extended to be analytic at z_1, then the radius of convergence of the Taylor series is at most $|z_1 - z_0|$, since a larger radius of convergence would imply that a possible domain of analyticity contains z_1.

proof after examples....

Examples

$e^{w} = \sum_{n=0}^{\infty} \frac{w^n}{n!}$

1) Find the Taylor series for $f(z) = e^{z^2}$ at $z_0 = 0$, and its radius of convergence.

$$\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$$

way better than using $\frac{f^{(n)}(z_0)}{n!}$!!

subs.

$(R = \infty)$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1. \qquad 1 + z +$$

2) Find the Taylor series for $f(z) = \dfrac{1}{(z-1)^2}$ at $z_0 = 0$, along with its radius of convergence.

$\cdot \ \frac{d}{dz}(1-z)^{-1} = -(1-z)^{-2}(-1) = \frac{1}{(z-1)^2}.$

diff

$$\frac{1}{(z-1)^2} = \sum_{n=1}^{\infty} n z^{n-1} \left(= \sum_{k=0}^{\infty} (k+1) z^k \right) \qquad |z| < 1$$

$k = n-1$

$k+1 = n$

3) Find the Taylor series for $f(z) = \log(1+z)$ at $z_0 = 0$, along with its radius of convergence.

$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n = \sum (-1)^n z^n$$

geometric

integrate!

to be continued ...

4) Find the Taylor series of $f(z) = \dfrac{1}{z^2 - z - 6} = \dfrac{1}{5}\left(\dfrac{1}{z-3} - \dfrac{1}{z+2}\right)$ at $z_0 = 0$, along with its radius of convergence.

5) Define $\log(z) = \ln|z| + i\,arg(z)$ on the branch domain $0 < arg(z) < 2\pi$. Find the Taylor series for $\log(z)$ at $z_0 = 1 + i$, and find the radius of convergence using the ratio test for absolute convergence. Explain why your answer may seem surprising at first.

Theorem 4 If f is analytic in $D(z_0; R)$ then the Taylor series for f at z_0,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

converges to f in $D(z_0; R)$. Thus the radius of convergence of the power series is at least R.

proof: Let $|z - z_0| \le r < R_1 < R$, $\gamma(t) = z_0 + R_1 e^{it}$, $0 \le t \le 2\pi$, the circle $\zeta - z_0 = R_1$.

Then the Cauchy integral formula reads

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

We use geometric series magic:

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)}\, d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)} \frac{1}{1 - \dfrac{(z - z_0)}{(\zeta - z_0)}}\, d\zeta$$

using the geometric series for $\dfrac{1}{1 - w}$ with $|w| \le \dfrac{r}{R_1}$:

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)} \sum_{n=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^n d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} f(\zeta)\frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}\, d\zeta.$$

Because $|f|$ is bounded on γ and

$$\frac{|z - z_0|^n}{|\zeta - z_0|^{n+1}} \le \frac{1}{R_1}\left(\frac{r}{R_1}\right)^n,$$

the series which is the integrand converges uniformly on γ so we may interchange the summation with the integration, (and then pull each $(z - z_0)^n$ through the integral:

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z - z_0)^n \int_\gamma \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta$$

$$= \sum_{n=0}^{\infty} (z - z_0)^n \quad \frac{1}{2\,\pi\, i} \int_{\gamma} \sum_{n=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

by the Cauchy integral formula for derivatives!

Q.E.D.